

$\mathbb{Z}/2$ -EQUIVARIANT AND \mathbb{R} -MOTIVIC STABLE STEMS

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ABSTRACT. We establish an isomorphism between the stable homotopy groups $\hat{\pi}_{s,w}^{\mathbb{R}}$ of the 2-completed \mathbb{R} -motivic sphere spectrum and the stable homotopy groups $\hat{\pi}_{s,w}^{\mathbb{Z}/2}$ of the 2-completed $\mathbb{Z}/2$ -equivariant sphere spectrum when $s \geq 3w - 5$ or $s \leq -1$.

1. INTRODUCTION

This paper is a sequel to [3], where we computed some of the stable homotopy groups of the 2-completed motivic sphere spectrum over the ground field \mathbb{R} . Here we explain that in a certain range these groups agree with the analogous $\mathbb{Z}/2$ -equivariant (but non-motivic) stable homotopy groups.

There is an equivariant realization functor from \mathbb{R} -motivic stable homotopy theory to $\mathbb{Z}/2$ -equivariant homotopy theory, induced by assigning to every scheme X over \mathbb{R} the associated analytic space $X(\mathbb{C})$ with complex conjugation [9, Section 3.3]. This induces a map

$$(1.1) \quad \hat{\pi}_{*,*}^{\mathbb{R}} \rightarrow \hat{\pi}_{*,*}^{\mathbb{Z}/2}$$

of bigraded rings, where the domain is the stable homotopy ring of the 2-completed \mathbb{R} -motivic sphere spectrum, and the target is the stable homotopy ring of the 2-completed $\mathbb{Z}/2$ -equivariant sphere spectrum. Each group $\hat{\pi}_{s,w}^{\mathbb{Z}/2}$ is finitely-generated, so the 2-completion on the right is very mild. The reader should beware that the stable homotopy groups of the 2-completed motivic sphere are not necessarily the same as the algebraic 2-completions of the stable homotopy groups of the uncompleted motivic sphere. One must account for η -completion as well [5, Theorem 1]. For the purposes of this paper, $\hat{\pi}_{*,*}^{\mathbb{R}}$ and $\hat{\pi}_{*,*}^{\mathbb{Z}/2}$ can be defined as the objects to which the \mathbb{R} -motivic and $\mathbb{Z}/2$ -equivariant Adams spectral sequences converge, respectively.

The $\mathbb{Z}/2$ -equivariant stable homotopy groups were computed in a range by Araki and Iriye [1, 6], although the method of computation and statements of results are difficult to navigate. A goal of the work begun in [3] is to better understand the Araki-Iriye results by lifting as much as possible back to \mathbb{R} -motivic homotopy theory via the map (1.1). The present paper demonstrates that this is possible in a range.

1.2. Equivariant homotopy groups. Recall that $\mathbb{R}^{1,1}$ denotes the real line with the sign representation of $\mathbb{Z}/2$, whereas $\mathbb{R}^{1,0}$ denotes the real line with the trivial representation. For $p \geq q$ one sets

$$\mathbb{R}^{p,q} = (\mathbb{R}^{1,0})^{\oplus(p-q)} \oplus (\mathbb{R}^{1,1})^{\oplus q},$$

and $S^{p,q}$ is the one-point compactification of $\mathbb{R}^{p,q}$. These are the bigraded $\mathbb{Z}/2$ -equivariant spheres, and we write $\pi_{p,q}^{\mathbb{Z}/2}$ for the $\mathbb{Z}/2$ -equivariant stable homotopy

group $[S^{p,q}, S^{0,0}]$. These groups were computed by Araki-Iriye in the range $p \leq 13$, although the calculations for $p = 12$ and $p = 13$ were announced without proof [1] [6].

One way to understand the global structure of $\pi_{*,*}^{\mathbb{Z}/2}$ is to break the calculation into pieces as follows. The **n th $\mathbb{Z}/2$ -equivariant Milnor-Witt stem** is the collection of groups

$$\oplus_p \pi_{p+n,p}^{\mathbb{Z}/2}.$$

The 0th Milnor-Witt stem is a subring of $\pi_{*,*}^{\mathbb{Z}/2}$, and the n th Milnor-Witt stem is a module over this subring.

Table 1 at the end of the article gives some partial results about $\mathbb{Z}/2$ -equivariant stable homotopy groups, arranged so that the groups in each row belong to a common Milnor-Witt stem.

We will give a global picture of the current knowledge of $\mathbb{Z}/2$ -equivariant stable homotopy groups. One piece of the global structure relates to the fixed-point map

$$\phi: \pi_{s,w}^{\mathbb{Z}/2} \rightarrow \pi_{s-w}$$

from the equivariant to the non-equivariant groups. This map is known to be split for $s \geq 2w$ [2, p. 284], and is an isomorphism for $s < 0$ [1, Proposition 7.0]. These splittings are represented by copies of π_{s-w} in Table 1.

The second piece of global structure consists of periodicity, for each fixed s , of the kernel of $\phi: \pi_{s,*} \rightarrow \pi_{s-*}$. Note that when $* > s$ this is $\pi_{s,*}$ itself, whereas when $s \geq 2w$ it is a summand (by the preceding paragraph). There are two difficulties with the periodicity phenomenon. First, the orders of the periodicities and the values of the periodic groups are rather complicated. See Table 2 of [6] for a complete description in the range $s \leq 13$, and beware that the indexing in that table differs from ours: the correspondence is given by the equations $s = p + q$ and $w = p$. Second, there are exceptions to the periodicity in the range $2w \geq s \geq w - 1$ [1, Proposition 4.8]. These exceptions are shown in red in Table 1. Note, however, that *some* of the groups in the range $2w \geq s \geq w - 1$ actually do assume the periodic values.

The groups $\pi_{p,0}^{\mathbb{Z}/2}$ and $\pi_{p,1}^{\mathbb{Z}/2}$ are also computed in [10] for $p \leq 13$ using the equivariant Adams spectral sequence based on Borel cohomology.

1.3. Motivic homotopy groups. The motivic setup [9] is similar to the equivariant setup. Now $S^{1,0}$ is the simplicial circle, $S^{1,1}$ is the scheme $\mathbb{A}^1 - 0$, and $S^{p,q}$ is the appropriate smash product of copies of $S^{1,0}$ and $S^{1,1}$. We use the same notation $S^{p,q}$ for motivic spheres and equivariant spheres. Equivariant realization sends one to the other, so this abuse of notation generally does not lead to confusion.

We write $\pi_{p,q}^{\mathbb{R}}$ for the \mathbb{R} -motivic stable homotopy group $[S^{p,q}, S^{0,0}]$. The **n th \mathbb{R} -motivic Milnor-Witt stem** is the collection of groups

$$\oplus_p \pi_{p+n,p}^{\mathbb{R}}.$$

As in the equivariant case, the 0th Milnor-Witt stem is a subring, and the n th Milnor-Witt stem is a module over the 0th Milnor-Witt stem. Morel's connectivity theorem [8] shows that the negative Milnor-Witt stems are zero. Moreover, Morel has calculated the 0th Milnor-Witt stem in terms of Milnor-Witt K -theory [7, Section 6].

Morel's calculation gives an explicit description of $\pi_{-1,-1}^{\mathbb{R}}$, but it turns out to be a complicated uncountable group. In order to carry out further calculations,

we find it convenient to work with the stable homotopy groups of the 2-completed \mathbb{R} -motivic sphere. One could complete at odd primes as well, but we do not address that here.

We will now set aside the \mathbb{R} -motivic stable homotopy ring $\pi_{*,*}^{\mathbb{R}}$, and instead work with the stable homotopy ring $\hat{\pi}_{*,*}^{\mathbb{R}}$ of the 2-completed \mathbb{R} -motivic sphere. This ring splits into Milnor-Witt stems as before. The 2-complete negative Milnor-Witt stems are still zero, and the 2-complete 0th Milnor-Witt stem can be easily described with generators and relations. Moreover, the first, second, and third Milnor-Witt stems have been completely described [3]. The authors have preliminary data on the n th Milnor-Witt stems for $n \leq 15$; these results will appear in a future article.

1.4. The comparison. The map (1.1) is not an isomorphism in general. We know that the negative \mathbb{R} -motivic Milnor-Witt stems vanish, whereas Table 1 shows that in the $\mathbb{Z}/2$ -equivariant context the negative Milnor-Witt stems are non-trivial. In the 0th Milnor-Witt stems, the map (1.1) is an isomorphism when $p \leq 4$ but not in general [1, Theorem 12.4(iii)]. Likewise, the computations of [3] show that $\hat{\pi}_{*,*}^{\mathbb{R}}$ vanishes in the first Milnor-Witt stem for weights larger than 2, whereas the $\mathbb{Z}/2$ -equivariant analog of this is false.

Nevertheless, we find that the map (1.1) is an isomorphism in a certain range. The following is the main result of the paper.

Theorem 1.5. *The realization map $\hat{\pi}_{s,w}^{\mathbb{R}} \rightarrow \hat{\pi}_{s,w}^{\mathbb{Z}/2}$ is an isomorphism in the range $s \geq 3w - 5$ or $s \leq -1$.*

In Table 1 the range from the above theorem is shaded. All of the groups in that region coincide, up to 2-completion, with their 2-completed \mathbb{R} -motivic analogues.

Example 1.6. We computed in [3] that $\hat{\pi}_{7,4}^{\mathbb{R}}$ contains an element of order 32. Theorem 1.5 implies that $\hat{\pi}_{7,4}^{\mathbb{Z}/2}$ also contains an element of order 32. This is somewhat surprising because the classical image of J in the 7-stem has order 16. In fact, this phenomenon is already apparent in the results of Araki and Iriye [1]. This observation calls strongly for a more careful study of the motivic and equivariant images of J .

We note two immediate consequences of Theorem 1.5. First, consider the map $\hat{\pi}_{s,w}^{\mathbb{R}} \rightarrow \hat{\pi}_{s-w}$ induced by taking fixed points of equivariant realization. Theorem 1.5 implies that this map is an isomorphism in the range $s \leq -1$ and a split surjection for $s \geq \max\{3w - 5, 2w\}$, based on the analogous facts for $\phi: \pi_{s,w}^{\mathbb{Z}/2} \rightarrow \pi_{s-w}$. Secondly, the known periodicity phenomena in the $\pi_{s,*}^{\mathbb{Z}/2}$ groups can now be transplanted into the \mathbb{R} -motivic context, as in Corollary 1.7.

Corollary 1.7. *For fixed s in the range $s \geq \max\{3w - 5, 2w\}$, the complementary summands of $\hat{\pi}_{s-w}$ in $\hat{\pi}_{s,w}^{\mathbb{R}}$ are periodic in w .*

We do not give the periods in Corollary 1.7, but specific formulas for these are known from the equivariant context.

Corollary 1.7 describes a qualitative property of \mathbb{R} -motivic stable homotopy groups that deserves further study and is related to τ^{2^n} -periodic families in the \mathbb{R} -motivic Adams spectral sequence (see [3] for an introduction to this basic phenomenon). We expect to return to the topic of motivic periodicity in future work.

The proof of Theorem 1.5 is straightforward. Equivariant realization induces a map from the \mathbb{R} -motivic Adams spectral sequence to the $\mathbb{Z}/2$ -equivariant Adams

spectral sequence. The \mathbb{R} -motivic and $\mathbb{Z}/2$ -equivariant Steenrod algebras agree in a range of dimensions. This gives an isomorphism on cobar complexes in a range, which shows that \mathbb{R} -motivic and $\mathbb{Z}/2$ -equivariant Ext groups agree in a range. In other words, the $\mathbb{Z}/2$ -equivariant and \mathbb{R} -motivic Adams E_2 -pages agree in a range. Finally, this induces an isomorphism in homotopy groups in a range. The only complications arise as matters of bookkeeping.

1.8. Notation. For the reader's convenience, we record here notation used in the article.

- $\mathbb{M}_2^{\mathbb{R}}$ is the \mathbb{R} -motivic homology of a point with \mathbb{F}_2 coefficients.
- $\mathbb{M}_2^{\mathbb{Z}/2}$ is the $\mathbb{Z}/2$ -equivariant homology of a point with \mathbb{F}_2 coefficients.
- $\mathcal{A}_{\mathbb{R}}$ is the dual \mathbb{R} -motivic Steenrod algebra. We grade elements in the form (t, w) , where t is the internal Steenrod degree and w is the motivic weight.
- $\mathcal{A}_{\mathbb{Z}/2}$ is the dual $\mathbb{Z}/2$ -equivariant Steenrod algebra. We grade elements in the form (t, w) , where t is the internal Steenrod degree and w is the equivariant weight.
- $\overline{\mathcal{A}}_{\mathbb{R}}$ is the augmentation ideal of $\mathcal{A}_{\mathbb{R}}$.
- $\overline{\mathcal{A}}_{\mathbb{Z}/2}$ is the augmentation ideal of $\mathcal{A}_{\mathbb{Z}/2}$.
- $C_{\mathbb{R}}^*$ is the \mathbb{R} -motivic cobar complex.
- $C_{\mathbb{Z}/2}^*$ is the $\mathbb{Z}/2$ -equivariant cobar complex.
- $\text{Ext}_{\mathbb{R}} = \text{Ext}_{\mathcal{A}_{\mathbb{R}}}(\mathbb{M}_2^{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}})$ is the cohomology of the \mathbb{R} -motivic Steenrod algebra. We grade elements in the form (s, f, w) , where $s = t - f$ is the stem, f is the Adams filtration, and w is the motivic weight.
- $\text{Ext}_{\mathbb{Z}/2} = \text{Ext}_{\mathcal{A}_{\mathbb{Z}/2}}(\mathbb{M}_2^{\mathbb{Z}/2}, \mathbb{M}_2^{\mathbb{Z}/2})$ is the cohomology of the $\mathbb{Z}/2$ -equivariant Steenrod algebra. We grade elements in the form (s, f, w) , where $s = t - f$ is the stem, f is the Adams filtration, and w is the equivariant weight.
- $\hat{\pi}_{*,*}^{\mathbb{R}}$ is the stable homotopy ring of the 2-completed \mathbb{R} -motivic sphere. We grade elements in the form (s, w) , where s is the stem and w is the motivic weight.
- $\hat{\pi}_{*,*}^{\mathbb{Z}/2}$ is the stable homotopy ring of the 2-completed $\mathbb{Z}/2$ -equivariant sphere. We grade elements in the form (s, w) , where s is the stem and w is the equivariant weight.

For sake of tradition, we refer to $\mathcal{A}_{\mathbb{R}}$ and $\mathcal{A}_{\mathbb{Z}/2}$ as Steenrod algebras. More precisely, $(\mathbb{M}_2^{\mathbb{R}}, \mathcal{A}_{\mathbb{R}})$ and $(\mathbb{M}_2^{\mathbb{Z}/2}, \mathcal{A}_{\mathbb{Z}/2})$ are Hopf algebroids, not Hopf algebras, because $\mathbb{M}_2^{\mathbb{R}}$ is a non-trivial $\mathcal{A}_{\mathbb{R}}$ -module, and $\mathbb{M}_2^{\mathbb{Z}/2}$ is a non-trivial $\mathcal{A}_{\mathbb{Z}/2}$ -module.

2. THE MOTIVIC AND EQUIVARIANT STEENROD ALGEBRAS

Let $H^{\mathbb{R}}$ denote the \mathbb{R} -motivic Eilenberg-MacLane spectrum representing motivic cohomology with \mathbb{F}_2 coefficients, and let $\mathbb{M}_2^{\mathbb{R}} = \pi_{*,*}(H^{\mathbb{R}})$ be the homology of a point. Recall that $\mathbb{M}_2^{\mathbb{R}}$ equals $\mathbb{F}_2[\tau, \rho]$ where τ has homological degree $(0, -1)$ and ρ has homological degree $(-1, -1)$ [11].

Let $\mathcal{A}_{\mathbb{R}} = \pi_{*,*}^{\mathbb{R}}(H^{\mathbb{R}} \wedge H^{\mathbb{R}})$ be the dual \mathbb{R} -motivic Steenrod algebra. Recall that $\mathcal{A}_{\mathbb{R}}$ is equal to

$$\mathbb{M}_2[\tau_0, \tau_1, \dots, \xi_0, \xi_1, \dots] / (\xi_0 = 1, \tau_k^2 = \tau \xi_{k+1} + \rho \tau_{k+1} + \rho \tau_0 \xi_{k+1}),$$

where ξ_i has bidegree $(2(2^i - 1), 2^i - 1)$ and τ_i has bidegree $(2^{i+1} - 1, 2^i - 1)$ [12]. For a summary of the complete Hopf algebroid structure, see [3, Section

2]. Observe that $\mathcal{A}_{\mathbb{R}}$ is free as a left $\mathbb{M}_2^{\mathbb{R}}$ -module, with basis given by monomials $\tau_0^{\epsilon_0} \tau_1^{\epsilon_1} \dots \tau_r^{\epsilon_r} \xi_1^{n_1} \dots \xi_s^{n_s}$ where $0 \leq \epsilon_i \leq 1$ and $n_i \geq 0$. We abbreviate such a monomial as $\tau^{\epsilon} \xi^n$.

When we build the cobar complex, we will use the augmentation ideal $\overline{\mathcal{A}}_{\mathbb{R}}$ of $\mathcal{A}_{\mathbb{R}}$, i.e., the kernel of the augmentation map $\mathcal{A}_{\mathbb{R}} \rightarrow \mathbb{M}_2^{\mathbb{R}}$. Observe that $\overline{\mathcal{A}}_{\mathbb{R}}$ is also free as a left $\mathbb{M}_2^{\mathbb{R}}$ -module, with the same basis as for $\mathcal{A}_{\mathbb{R}}$ except that the monomial 1 is excluded.

Similarly, let $H^{\mathbb{Z}/2}$ denote the $\mathbb{Z}/2$ -equivariant Eilenberg-MacLane spectrum corresponding to the constant Mackey functor with value \mathbb{F}_2 . Write $\mathbb{M}_2^{\mathbb{Z}/2} = \pi_{*,*}(H^{\mathbb{Z}/2})$ for the coefficient ring and $\mathcal{A}_{\mathbb{Z}/2} = \pi_{*,*}(H^{\mathbb{Z}/2} \wedge H^{\mathbb{Z}/2})$ for the $\mathbb{Z}/2$ -equivariant dual Steenrod algebra.

We will now recall an explicit description of $\mathbb{M}_2^{\mathbb{Z}/2}$ [4, Proposition 6.2]. It contains $\mathbb{M}_2^{\mathbb{R}}$ as a subring, but also contains a “dual copy” in opposing dimensions. Figure 2 gives a complete description of $\mathbb{M}_2^{\mathbb{Z}/2}$. Every dot denotes a copy of \mathbb{F}_2 , vertical lines represent multiplication by τ , and diagonal lines represent multiplication by ρ .

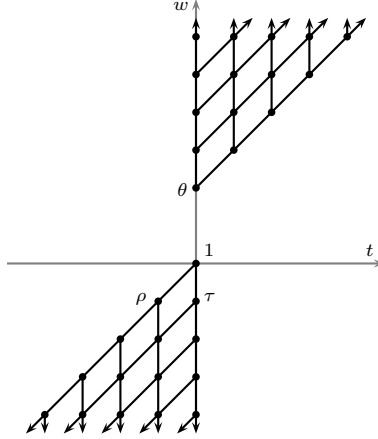


FIGURE 1. The equivariant coefficient ring $\mathbb{M}_2^{\mathbb{Z}/2}$ (homological grading)

In words, $\mathbb{M}_2^{\mathbb{Z}/2}$ in bidegree (t, w) consists of a copy of \mathbb{F}_2 when:

- (1) $t \geq 0$ and $w \geq t + 2$, or
- (2) $t \leq 0$ and $w \leq t$.

The element in bidegree $(0, 2)$ is called θ , and the other elements in the “dual copy” are typically named $\frac{\theta}{\tau^k \rho^l}$ for $k \geq 0$ and $l \geq 0$. This naming convention respects the product structure, although one must remember that neither τ nor ρ is actually invertible. Any two elements of the form $\frac{\theta}{\tau^k \rho^l}$ multiply to zero. These details about the product structure will not be needed in our analysis.

The dual $\mathbb{Z}/2$ -equivariant Steenrod algebra $\mathcal{A}_{\mathbb{Z}/2}$ has the same description as the \mathbb{R} -motivic Steenrod algebra, but with $\mathbb{M}_2^{\mathbb{R}}$ replaced with $\mathbb{M}_2^{\mathbb{Z}/2}$ [4, Theorem 6.41]. In particular, note that $\mathcal{A}_{\mathbb{Z}/2}$ is free as a left $\mathbb{M}_2^{\mathbb{Z}/2}$ -module on the same basis. More explicitly, $\mathcal{A}_{\mathbb{Z}/2}$ is equal to $\mathbb{M}_2^{\mathbb{Z}/2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \mathcal{A}_{\mathbb{R}}$. The augmentation ideal $\overline{\mathcal{A}}_{\mathbb{Z}/2}$ of $\mathcal{A}_{\mathbb{Z}/2}$ is also free as a left $\mathbb{M}_2^{\mathbb{Z}/2}$ -module, with the same basis as for $\mathcal{A}_{\mathbb{Z}/2}$ except that the monomial 1 is excluded, so $\overline{\mathcal{A}}_{\mathbb{Z}/2}$ is equal to $\mathbb{M}_2^{\mathbb{Z}/2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \overline{\mathcal{A}}_{\mathbb{R}}$.

Equivariant realization from \mathbb{R} -motivic homotopy theory to $\mathbb{Z}/2$ -equivariant homotopy theory yields a map $(\mathcal{A}_{\mathbb{R}}, \mathbb{M}_2^{\mathbb{R}}) \rightarrow (\mathcal{A}_{\mathbb{Z}/2}, \mathbb{M}_2^{\mathbb{Z}/2})$ of Hopf algebroids. This map is just the evident inclusion of $\mathbb{M}_2^{\mathbb{R}}$ into $\mathbb{M}_2^{\mathbb{Z}/2}$, and of $\mathcal{A}_{\mathbb{R}}$ into $\mathcal{A}_{\mathbb{Z}/2}$.

Lemma 2.1. *Let $\tau^{\epsilon}\xi^n$ have bidegree (t, w) . Then $t \leq 3w + 1$.*

Proof. The bidegree of each ξ_i satisfies the inequality $t \leq 3w$. Similarly, if $i \geq 1$, then the bidegree of τ_i also satisfies $t \leq 3w$. Therefore the bidegree of $\tau^{\epsilon}\xi^n$ satisfies the inequality $t \leq 3w$ if $\epsilon_0 = 0$.

On the other hand, if $\epsilon_0 = 1$, then write $\tau^{\epsilon}\xi^n$ as $\tau_0\tau^{\epsilon'}\xi^n$, where $\epsilon'_0 = 0$. The bidegree of $\tau^{\epsilon'}\xi^n$ satisfies the inequality $t \leq 3w$, so the bidegree of $\tau_0\tau^{\epsilon'}\xi^n$ satisfies $t \leq 3w + 1$. \square

Remark 2.2. In fact, one can make a much stronger statement about the bidegrees of the elements $\tau^{\epsilon}\xi^n$. In general, the bidegree of such an element satisfies the inequality $t \geq 2^{c+1} - c - 2$, where $c = t - 2w$ is the ‘‘Chow degree’’. However, this stronger inequality does not end up yielding a stronger result about stable homotopy groups. Likewise, the result in the following lemma is non-optimal—but the slope of 3 is chosen precisely because it interacts well with the bound from the previous lemma.

Lemma 2.3. *Let (t, w) be the bidegree of the element $\frac{\theta}{\rho^{\kappa}\tau^t}$ in $\mathbb{M}_2^{\mathbb{Z}/2}$. Then one has $t \leq 3w - 6$.*

Proof. In Figure 2, the elements of the form $\frac{\theta}{\rho^{\kappa}\tau^t}$ all lie on or above the line $t = 3w - 6$. \square

3. COBAR COMPLEXES AND EXT GROUPS

Next we proceed to the cobar complexes of $\mathcal{A}_{\mathbb{R}}$ and $\mathcal{A}_{\mathbb{Z}/2}$, respectively. These cobar complexes are differential graded algebras whose homologies give the \mathbb{R} -motivic and $\mathbb{Z}/2$ -equivariant Ext groups. We will obtain an isomorphism of Ext groups in a range by establishing an isomorphism of cobar complexes in a range.

Let $C_{\mathbb{R}}^*$ and $C_{\mathbb{Z}/2}^*$ be the \mathbb{R} -motivic and $\mathbb{Z}/2$ -equivariant cobar complexes. By definition, $C_{\mathbb{R}}^f$ is equal to

$$\overline{\mathcal{A}}_{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \overline{\mathcal{A}}_{\mathbb{R}} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \cdots \otimes_{\mathbb{M}_2^{\mathbb{R}}} \overline{\mathcal{A}}_{\mathbb{R}},$$

where there are f factors in the tensor product. Similarly, $C_{\mathbb{Z}/2}^f$ is equal to

$$\overline{\mathcal{A}}_{\mathbb{Z}/2} \otimes_{\mathbb{M}_2^{\mathbb{Z}/2}} \overline{\mathcal{A}}_{\mathbb{Z}/2} \otimes_{\mathbb{M}_2^{\mathbb{Z}/2}} \cdots \otimes_{\mathbb{M}_2^{\mathbb{Z}/2}} \overline{\mathcal{A}}_{\mathbb{Z}/2}.$$

Lemma 3.1. *The $\mathbb{Z}/2$ -equivariant cobar complex $C_{\mathbb{Z}/2}^*$ is isomorphic to $\mathbb{M}_2^{\mathbb{Z}/2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} C_{\mathbb{R}}^*$.*

Proof. Use that $\overline{\mathcal{A}}_{\mathbb{Z}/2}$ is equal to $\mathbb{M}_2^{\mathbb{Z}/2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} \overline{\mathcal{A}}_{\mathbb{R}}$. Then $C_{\mathbb{Z}/2}^f$ can be rewritten as $\mathbb{M}_2^{\mathbb{Z}/2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} C_{\mathbb{R}}^f$. \square

Lemma 3.2. *The map $C_{\mathbb{R}}^f \rightarrow C_{\mathbb{Z}/2}^f$ is:*

- an injection in all degrees.
- an isomorphism in degrees satisfying $t - f \geq 3w - 5$.
- an isomorphism in degrees satisfying $t \leq f - 1$.

Proof. By Lemma 3.1, we are considering the obvious map $C_{\mathbb{R}}^f \rightarrow \mathbb{M}_2^{\mathbb{Z}/2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} C_{\mathbb{R}}^f$. Since the map $\mathbb{M}_2^{\mathbb{R}} \rightarrow \mathbb{M}_2^{\mathbb{Z}/2}$ is injective and $C_{\mathbb{R}}^f$ is free over $\mathbb{M}_2^{\mathbb{R}}$, it follows that the map $C_{\mathbb{R}}^f \rightarrow \mathbb{M}_2^{\mathbb{Z}/2} \otimes_{\mathbb{M}_2^{\mathbb{R}}} C_{\mathbb{R}}^f$ is injective for every $f \geq 0$.

Consider a typical element $\frac{\theta}{\rho^k \tau^l} [\zeta_1 | \cdots | \zeta_f]$ of the cokernel of the map. By Lemma 2.1, each ζ_i has bidegree (t_i, w_i) satisfying $t_i \leq 3w_i + 1$. Summing over i , we obtain that $[\zeta_1 | \cdots | \zeta_f]$ has bidegree satisfying $t \leq 3w + f$. Finally, Lemma 2.3 implies that the bidegree of $\frac{\theta}{\rho^k \tau^l} [\zeta_1 | \cdots | \zeta_f]$ satisfies $t \leq 3w + f - 6$. Therefore, the cokernel vanishes in bidegrees satisfying $t - f \geq 3w - 5$.

Similarly, each ζ_i has bidegree (t_i, w_i) satisfying $t_i \geq 1$, so $[\zeta_1 | \cdots | \zeta_f]$ has bidegree satisfying $t \geq f$. Then the bidegree of $\frac{\theta}{\rho^k \tau^l} [\zeta_1 | \cdots | \zeta_f]$ also satisfies $t \geq f$. Therefore, the cokernel vanishes in bidegrees satisfying $t \leq f - 1$. \square

Remark 3.3. The inequalities in Lemma 3.2 are sharp in the following sense. The element $\theta[\tau_0 \tau_1 | \tau_0 \tau_1 | \cdots | \tau_0 \tau_1]$ of $C_{\mathbb{Z}/2}^f$ lies on the line $t - f = 3w - 6$, and it does not belong to the image of $C_{\mathbb{R}}^f$. Also, the element $\theta[\tau_0 | \tau_0 | \cdots | \tau_0]$ of $C_{\mathbb{Z}/2}^f$ lies on the line $t = f$, and it does not belong to the image of $C_{\mathbb{R}}^f$.

The following lemma from homological algebra will let us deduce an Ext isomorphism from the cobar isomorphism of Lemma 3.2. The two parts are dual, and the proofs are simple diagram chases.

Lemma 3.4. *Let $C_* \rightarrow D_*$ be a map of homologically graded chain complexes (so the differentials decrease degree).*

- (a) *Suppose that $C_i \rightarrow D_i$ is an isomorphism for all $i \geq n + 1$, and an injection for $i = n$. Then the map $H_i(C) \rightarrow H_i(D)$ of homology groups is:*
 - *an injection for $i = n$,*
 - *an isomorphism for $i \geq n + 1$.*
- (b) *Dually, suppose that $C_i \rightarrow D_i$ is an isomorphism for all $i \leq n - 1$, and a surjection for $i = n$. Then the map $H_i(C) \rightarrow H_i(D)$ of homology groups*
 - *is an isomorphism for $i \leq n - 1$,*
 - *a surjection for $i = n$.*

We will use the grading (s, f, w) for Ext groups, where s is the stem, f is the Adams filtration, and w is the weight. An element of degree (s, f, w) occurs at Cartesian coordinates (s, f) in a standard Adams chart. Recall that $s = t - f$, where t is the internal Steenrod degree.

Proposition 3.5. *In degree (s, f, w) , the map $\text{Ext}_{\mathbb{R}} \rightarrow \text{Ext}_{\mathbb{Z}/2}$ is:*

- *an injection if $s = 3w - 6$.*
- *an isomorphism if $s \geq 3w - 5$.*

Proof. The claims follow immediately from Lemmas 3.2 and 3.4 because Ext can be computed as the homology of the cobar construction. \square

Proposition 3.6. *In degree (s, f, w) , the map $\text{Ext}_{\mathbb{R}} \rightarrow \text{Ext}_{\mathbb{Z}/2}$ is an isomorphism if $s \leq -1$.*

Proof. Lemmas 3.2 and 3.4 imply that the map is an isomorphism if $s \leq -2$ and is a surjection if $s = -1$. In order to obtain the isomorphism for $s = -1$, we need to investigate the cobar complex a little further.

In degrees satisfying $s = 0$, i.e., $t = f$, the cokernel of the map $C_{\mathbb{R}}^* \rightarrow C_{\mathbb{Z}/2}^*$ of cobar complexes consists of elements of the form $\frac{\theta}{\tau^a}[\tau_0|\tau_0|\cdots|\tau_0]$. All of these elements are cycles in the $\mathbb{Z}/2$ -equivariant cobar complex. A diagram chase now shows that $\text{Ext}_{\mathbb{R}} \rightarrow \text{Ext}_{\mathbb{Z}/2}$ is an isomorphism if $s = -1$. \square

The following finiteness condition for $\text{Ext}_{\mathbb{R}}$ implies that there are only finitely many Adams differentials in any given degree. We will need this fact in Section 4 when we analyze the Adams spectral sequence.

Lemma 3.7. *In each degree (s, f, w) , the group $\text{Ext}_{\mathbb{R}}^{(s, f, w)}$ is a finite-dimensional \mathbb{F}_2 -vector space.*

Proof. As described in [3, Section 3], there is a ρ -Bockstein spectral sequence converging to $\text{Ext}_{\mathbb{R}}$. It suffices to show that the E_1 -page of this spectral sequence is finite-dimensional over \mathbb{F}_2 in each tridegree. In degree (s, f, w) , this E_1 -page consists of elements of the form $\rho^k x$, where $k \geq 0$ and x belongs to the \mathbb{C} -motivic Ext group in degree $(s + k, f, w + k)$.

The \mathbb{C} -motivic Ext groups have a vanishing plane, as described in [3, Lemma 2.2]. In this case, the vanishing plane implies that $k \leq s + f - 2w$ if x is non-zero. Since k is non-negative this means there are only finitely-many values of k that contribute to the E_1 -page of our spectral sequence in degree (s, f, w) .

Finally, the \mathbb{C} -motivic Ext groups are degreewise finite-dimensional. This follows from the fact that the E_1 -page of the motivic May spectral sequence is degreewise finite-dimensional. \square

4. HOMOTOPY GROUPS

We now come to our main results comparing \mathbb{R} -motivic and $\mathbb{Z}/2$ -equivariant homotopy groups.

Theorem 4.1. *The map $\hat{\pi}_{s,w}^{\mathbb{R}} \rightarrow \hat{\pi}_{s,w}^{\mathbb{Z}/2}$ is:*

- *an injection if $s = 3w - 6$.*
- *an isomorphism if $s \geq 3w - 5$.*

Proof. Proposition 3.5 gives an isomorphism (in a range) between the E_2 -pages of the \mathbb{R} -motivic and $\mathbb{Z}/2$ -equivariant Adams spectral sequences. Inductively, Lemma 3.4 gives isomorphisms (in a range) between the E_r -pages of the spectral sequences for all r . The finiteness condition of Lemma 3.7 guarantees that for each degree (s, f, w) , there exists an r such that the E_{∞} -page is isomorphic to the E_r -page. Therefore, we obtain an isomorphism of E_{∞} -pages in a range.

The E_{∞} -pages are associated graded objects of the stable homotopy groups. This implies that the stable homotopy groups are isomorphic as well.

The same style of argument applies to the claim about injections. \square

Theorem 4.2. *The map $\hat{\pi}_{s,w}^{\mathbb{R}} \rightarrow \hat{\pi}_{s,w}^{\mathbb{Z}/2}$ is an isomorphism if $s \leq -1$.*

Proof. The argument from Theorem 4.1 implies that the map is an isomorphism for $s \leq -2$ and a surjection for $s = -1$. In order to obtain the isomorphism for $s = -1$, we have to investigate the Adams E_2 -pages slightly further.

Recall from the proof of Proposition 3.6 that in degrees satisfying $s = 0$, the cokernel of the map $C_{\mathbb{R}}^* \rightarrow C_{\mathbb{Z}/2}^*$ of cobar complexes consists of elements of the form $\frac{\theta}{\tau^k}[\tau_0|\tau_0|\cdots|\tau_0]$. Therefore, in degrees satisfying $s = 0$, the cokernel of the

map $\text{Ext}_{\mathbb{R}} \rightarrow \text{Ext}_{\mathbb{Z}/2}$ consists of elements of the form $\frac{\theta}{\tau^k} h_0^i$. These elements are all permanent cycles in the $\mathbb{Z}/2$ -equivariant Adams spectral sequence. In other words, there is a one-to-one correspondence between \mathbb{R} -motivic and $\mathbb{Z}/2$ -equivariant Adams differentials from the 0-stem to the (-1) -stem.

A diagram chase now establishes that the \mathbb{R} -motivic and $\mathbb{Z}/2$ -equivariant E_{∞} -pages are isomorphic for $s = -1$. This passes to an isomorphism of stable homotopy groups. \square

We restate Theorem 4.1 in an equivalent form that is useful from the Milnor-Witt degree perspective.

Corollary 4.3. *On the n th Milnor-Witt stems, the map $\hat{\pi}_{*,*}^{\mathbb{R}} \rightarrow \hat{\pi}_{*,*}^{\mathbb{Z}/2}$ is:*

- an isomorphism in stem s if $2s \leq 3n + 5$.
- an injection in stem s if $2s = 3n + 6$.

Proof. This is a straightforward algebraic rearrangement of Theorem 4.1, using that $n = s - w$. \square

5. EQUIVARIANT STABLE HOMOTOPY GROUPS

Table 1 summarizes some of the calculations of Araki and Iriye [1, 6]. The indices across the top indicate the stem s , while the indices at the left indicate the Milnor-Witt degree $s - w$. The \mathbb{R} -motivic and $\mathbb{Z}/2$ -equivariant stable homotopy groups are isomorphic in the shaded region, as described in Theorem 1.5.

For compactness, we use the following notation to indicate abelian groups:

- (1) $\infty = \mathbb{Z}$.
- (2) $n = \mathbb{Z}/n$.
- (3) $n \cdot m = \mathbb{Z}/n \oplus \mathbb{Z}/m$.
- (4) $n^k = (\mathbb{Z}/n)^k$.

The symbols π_k indicate that the classical stable homotopy group π_k splits via the fixed point map.

Table 1 is a companion to [6, Table 2], which gives the values of the periodic summands. The red symbols in Table 1 are exceptions to the periodicity.

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TABLE 1. Some values of $\pi_{s,w}^{\mathbb{Z}/2}$

| $s \backslash w$ | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|------------------|---------|---------|----------------------|-------------------|-------------------|--------------------------|------------------|------------------|--------------------------|
| 7 | π_7 | π_7 | $2 \cdot \pi_7$ | $4 \cdot \pi_7$ | $8 \cdot \pi_7$ | $48 \cdot 4 \cdot \pi_7$ | $16 \cdot \pi_7$ | $16 \cdot \pi_7$ | $16 \cdot 2 \cdot \pi_7$ |
| 6 | π_6 | π_6 | $\infty \cdot \pi_6$ | $2 \cdot \pi_6$ | $2^2 \cdot \pi_6$ | $2^2 \cdot \pi_6$ | $2 \cdot \pi_6$ | $2 \cdot \pi_6$ | $2^2 \cdot \pi_6$ |
| 5 | π_5 | π_5 | $2 \cdot \pi_5$ | $2^2 \cdot \pi_5$ | $2 \cdot \pi_5$ | $12 \cdot \pi_5$ | π_5 | π_5 | π_5 |
| 4 | π_4 | π_4 | $\infty \cdot \pi_4$ | π_4 | π_4 | $2 \cdot \pi_4$ | $2 \cdot \pi_4$ | $2 \cdot \pi_4$ | $4 \cdot \pi_4$ |
| 3 | π_3 | π_3 | $2 \cdot \pi_3$ | $4 \cdot \pi_3$ | $8 \cdot \pi_3$ | $24 \cdot 8 \cdot \pi_3$ | $8 \cdot \pi_3$ | $8 \cdot \pi_3$ | $8 \cdot \pi_3$ |
| 2 | π_2 | π_2 | $\infty \cdot \pi_2$ | $2 \cdot \pi_2$ | $2^2 \cdot \pi_2$ | $2 \cdot \pi_2$ | π_2 | 2 | 2 |
| 1 | π_1 | π_1 | $2 \cdot \pi_1$ | $2^2 \cdot \pi_1$ | $2 \cdot \pi_1$ | 24 | 0 | 0 | 0 |
| 0 | π_0 | π_0 | $\infty \cdot \pi_0$ | ∞ | ∞ | ∞ | ∞ | ∞ | $\infty \cdot 2$ |
| -1 | 0 | 0 | 0 | 0 | 0 | 12 | 0 | 0 | 2 |
| -2 | 0 | 0 | ∞ | 2 | 2^2 | 2^2 | 2 | 2 | 2^2 |
| -3 | 0 | 0 | 2 | 2^2 | 2 | 12 | 0 | 0 | 0 |

| | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|----|------------------------------------|--------------------------|--------------------------|---------------------------|----------------------------|-------------------|--------------|
| 7 | $240 \cdot 16 \cdot 2 \cdot \pi_7$ | $16 \cdot 2 \cdot \pi_7$ | $16 \cdot 2 \cdot \pi_7$ | $16 \cdot \pi_7$ | $2016 \cdot 4 \cdot \pi_7$ | $16 \cdot \pi_7$ | 48 |
| 6 | $2 \cdot \pi_6$ | $4 \cdot \pi_6$ | $4 \cdot 2 \cdot \pi_6$ | $6 \cdot 2^2 \cdot \pi_6$ | $2^2 \cdot \pi_6$ | $2^2 \cdot \pi_6$ | 2^2 |
| 5 | $240 \cdot \pi_5$ | $2^3 \cdot \pi_5$ | $2^5 \cdot \pi_5$ | $2^2 \cdot \pi_5$ | 504 | 0 | 3 |
| 4 | $2^2 \cdot \pi_4$ | $2^4 \cdot \pi_4$ | 2^2 | 3 | 0 | 0 | 0 |
| 3 | $480 \cdot 12 \cdot 4$ | $24 \cdot 4$ | $24 \cdot 2$ | 24 | $504 \cdot 24$ | 24 | $24 \cdot 3$ |
| 2 | 0 | 0 | 2 | $6 \cdot 2$ | 2 | 0 | 0 |
| 1 | 240 | 2^3 | 2^6 | 2^3 | $504 \cdot 2$ | 2 | 6 |
| 0 | $\infty \cdot 2^2$ | $\infty \cdot 2^4$ | $\infty \cdot 2^2$ | $\infty \cdot 3$ | ∞ | ∞ | ∞ |
| -1 | $120 \cdot 2$ | 2 | 2 | 0 | 252 | 0 | 3 |
| -2 | 2^2 | 4^2 | $8 \cdot 4 \cdot 2$ | $24 \cdot 2^3$ | $16 \cdot 2^2$ | $16 \cdot 2$ | $16 \cdot 2$ |
| -3 | 240 | 2^3 | 2^5 | 2^2 | 504 | 0 | 3 |

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